

Monday, September 15, Lecture 8: Change-of-basis & Diagonalization

Last time...

- A choice of basis $\mathcal{B} = \{v_1, \dots, v_n\}$ for a vector space V gives coordinates: if $w = a_1 v_1 + \dots + a_n v_n$,

$$[w]_{\mathcal{B}} = \begin{bmatrix} a_1 \\ \vdots \\ a_n \end{bmatrix}$$

- Given a linear map $f: V \rightarrow W$, we may represent it by a matrix $[f]$ by picking
 - bases for V and W .
 - If you change basis, this changes the matrix

Let $T: V \rightarrow V$ be linear.

There is a basis of V consisting of eigenvectors $v_1, \dots, v_n$ of T .



The matrix $[T]$ is diagonalizable.

Change-of-basis (linear change-of-coordinates)

• The correspondence

$$\begin{array}{c} \text{linear maps} \\ T: V \rightarrow W \end{array} \quad \Leftrightarrow \quad \begin{array}{c} \text{m} \times \text{n matrices} \\ [T] \end{array}$$

depends on bases for both V and W .

• We will mostly be thinking about linear operators $T: V \rightarrow V$.

• Typically, to avoid confusion, we fix a shared basis for the input and output.

• These are active transformations.

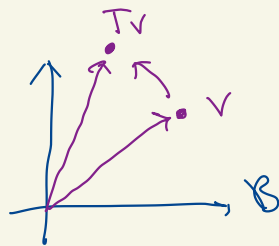
• Change-of-basis is the one case where we don't do this,

• These are passive transformations.

• Imagine $v \in V$ represents a particle, and the invertible matrix T is a rotation of the system.

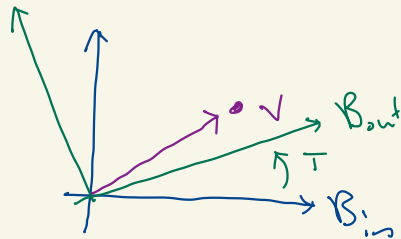
① Active transformation

Here, we think of T as rotating the vectors $v \mapsto Tv$ in a fixed reference basis B .



② Passive transformation

Here, we think of v as fixed while we rotate the reference basis $B_{in} \mapsto B_{out}$.



Notice: these have opposite effects for the same matrix T !

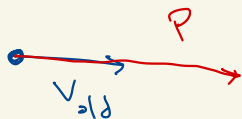
Ex: Let $V = \mathbb{R}$ represent position, and suppose

we choose the basis vector $v_{old} = 1 \text{ m}$.

Let $p \in V$ be the position of a particle, at 2 m.

we may find coordinates:

$$p = 2 v_{old} \Rightarrow [p]_{old} = [2].$$



Now, take a new basis $v_{new} = 10 \text{ cm} = \frac{1}{10} \text{ m}$.

Then the new coordinates are

$$p = 20 v_{new} \Rightarrow [p]_{new} = [20].$$



Notice: while $v_{new} = \frac{1}{10} v_{old}$, the change in coordinates of p is

$$[p]_{new} = [10] [p]_{old}$$

↑
this is the change-of-basis matrix.
notice it is the inverse of what
happens to basis vectors.

$$[10] = \left[\frac{1}{10}\right]^{-1}$$

Def: Change-of-basis

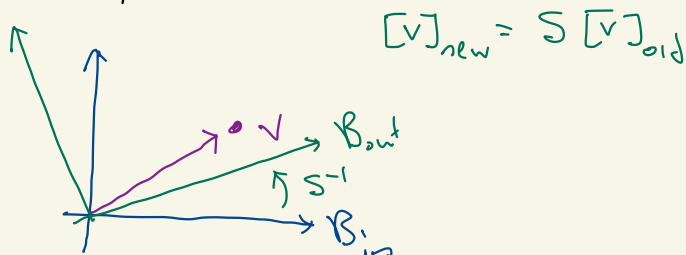
Let $\mathcal{B}_{\text{old}} = \{e_1, \dots, e_n\}$ and $\mathcal{B}_{\text{new}} = \{f_1, \dots, f_n\}$ be bases of a vector space V . The change-of-basis S from $\mathcal{B}_{\text{old}}$ to $\mathcal{B}_{\text{new}}$ is the invertible matrix whose columns are the coordinates of e_i 's in the new basis;

$$S = \begin{pmatrix} | & & | \\ [e_1]_{\text{new}} & \dots & [e_n]_{\text{new}} \\ | & & | \end{pmatrix}$$

It has the property that it transforms old coordinates into new ones:

$$[v]_{\text{new}} = S [v]_{\text{old}}$$

and it represents a passive transformation:



$$[v]_{\text{new}} = S [v]_{\text{old}}$$

- Notice: the backwards change-of-basis, $\mathcal{B}_{\text{new}}$ to $\mathcal{B}_{\text{old}}$, is the matrix S^{-1} , since

$$S^{-1} [v]_{\text{new}} = S^{-1} S [v]_{\text{old}} = [v]_{\text{old}}$$

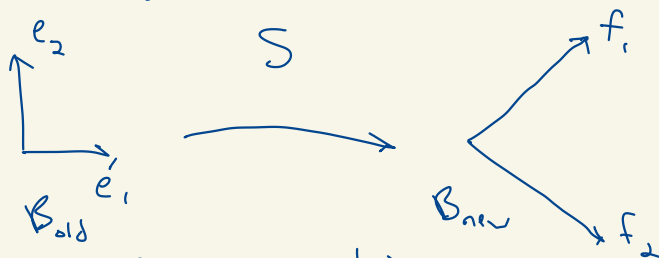
- Warning: some texts have the opposite convention, where

$$[v]_{\text{old}} = R [v]_{\text{new}}$$

It doesn't really matter - their R is our S^{-1} . Just be clear.

Ex:

Let $B_{old} = \{e_1 = (1, 0), e_2 = (0, 1)\}$, and $B_{new} = \{f_1 = (1, 1), f_2 = (1, -1)\}$
Then the change-of-basis looks like



The change-of-basis matrix is

$$S = \begin{pmatrix} | & | \\ [e_1]_{new} & [e_2]_{new} \\ | & | \end{pmatrix}$$

$$e_1 = \frac{1}{2}(f_1 + f_2) \Rightarrow [e_1]_{new} = \frac{1}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$e_2 = \frac{1}{2}(f_1 - f_2) \Rightarrow [e_2]_{new} = \frac{1}{2} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

so

$$S = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

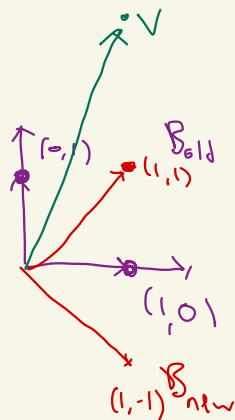
Check: $v = (3, 1)$. Then

$$v = 3e_1 + e_2 \Rightarrow [v]_{old} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

$$v = 2f_1 + f_2 \Rightarrow [v]_{new} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$[v]_{new} = S [v]_{old}$$

$$\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$



We can use this to change bases for matrices!

Let $T: V \rightarrow V$ be a linear operator and pick two bases $\mathcal{B}_{\text{old}} = \{e_1, \dots, e_n\}$ and $\mathcal{B}_{\text{new}} = \{f_1, \dots, f_n\}$, and let S take us $\mathcal{B}_{\text{old}} \rightarrow \mathcal{B}_{\text{new}}$.

Suppose we have $[T]_{\text{old}}$.

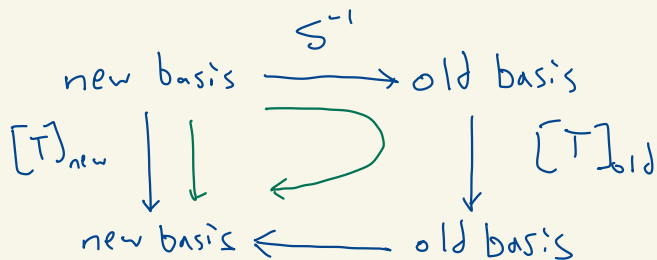
Then the matrix of T in the new basis is

$$[T]_{\text{new}} = S [T]_{\text{old}} S^{-1} \quad \leftarrow \text{hey... this looks familiar!}$$

Check: if we have a vector v , its coordinates are $[v]_{\text{new}}$

$$\begin{aligned} [T]_{\text{new}} [v]_{\text{new}} &= S [T]_{\text{old}} S^{-1} [v]_{\text{new}} \\ &= S [T]_{\text{old}} [v]_{\text{old}} \\ &= S [Tv]_{\text{old}} \\ &= [Tv]_{\text{new}} \end{aligned}$$

One way to think about this:



Let's revisit our example from last time.

Ex: $T: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is the linear map $T(x, y) = (x+y, x+y)$

$$\mathcal{B}_{st} = \{e_1 = (1, 0), e_2 = (0, 1)\}$$

$$\mathcal{B}_{eig} = \{f_1 = (1, 1), f_2 = (1, -1)\}$$

Then S which takes $\mathcal{B}_{st} \rightarrow \mathcal{B}_{eig}$ is

$$S = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

and S^{-1} (which takes $\mathcal{B}_{eig} \rightarrow \mathcal{B}_{st}$) is

$$S^{-1} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$\left[\text{check: } SS^{-1} = S^{-1}S = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right].$$

Then

$$\begin{aligned} [T]_{eig} &= S [T]_{st} S^{-1} \\ &= \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \\ &= \begin{pmatrix} 2 & 0 \\ 0 & 0 \end{pmatrix} \end{aligned}$$

- A big question: when are linear maps diagonalizable?
- Another: in physics and in life we have notions of length and angle. where is it now?
- The answers to both lie in Hilbert Spaces, vector spaces with an inner product.